# Scalability and Speedup Analysis

## Amdahl's Law: Fixed Problem Size

$$W = aW + (1-a)W$$

| | |
|---|---|
| $a$ | sequential execution |
| $(1-a)$ | parallel execution |

Fixed Problem Size Work Per Processor

$$W_1 = W$$

$$W_n = aW + (1-a)\frac{W}{n}$$

$$\frac{T_n}{W_n} = \frac{T_1}{W_1}$$

$$S_n = \frac{T_1}{T_n} = \frac{W_1}{W_n} = \frac{W}{aW + (1-a)\frac{W}{n}}$$

$$S_n = \frac{T_1}{T_n} = \frac{W_1}{W_n} = \frac{n}{na + (1-a)} = \frac{n}{1 + (n-1)a}$$

$$S_{n\to\infty} \Rightarrow \frac{n}{na} = \frac{1}{a}$$

Adding Overhead

$$S_n = \frac{T_1}{T_n} = \frac{W_1}{W_n} = \frac{W}{aW + (1-a)\frac{W}{n} + T_0} = \frac{1}{a + \frac{(1-a)}{n} + \frac{T_0}{W}}$$

$$S_{n\to\infty} = \frac{n}{1 - a + na + n T_0/W} \Rightarrow \frac{1}{a + T_0/W}$$

**Gustafson's Law: Fixed Time**

Fixed Time

Originally: $W_{ref} = W$

With n Nodes in the Fixed Time: $T_n = W \cdot k$ and $W_{new} = aW + (1-a)nW$

Equivalent Sequential Time for New Workload: $T_1 = W_{new} \cdot k = [aW + (1-a)nW] \cdot k$

$$S_n = \frac{T_1}{T_n} = \frac{W_{new} \cdot k}{W \cdot k} = \frac{aW + (1-a)nW}{W} = a + (1-a)n$$

$$S_{n\to\infty} = a + (1-a)n \Rightarrow (1-a)n$$

Adding Overhead

$$S_n = \frac{T_1}{T_n} = \frac{aW + (1-a)nW}{W + T_0} = \frac{a + (1-a)n}{1 + T_0/W}$$

$$S_{n\to\infty} = \frac{a + (1-a)n}{1 + T_0/W} \Rightarrow \frac{(1-a)n}{1 + T_0/W}$$

## Sun and Ni's Law: Memory Bounding

Fixed Time

$$\text{Originally: } W = aW + (1-a)W$$

With n Nodes and scaled workload: $W_n = aW + (1-a)\cdot G(n)\cdot \frac{W}{n}$ and $T_n = W_n \cdot k$

Equivalent Sequential Time: $W_1 = aW + (1-a)\cdot G(n)\cdot W$ and $T_1 = W_1 \cdot k$

$$S_n = \frac{T_1}{T_n} = \frac{W_1 \cdot k}{W_n \cdot k} = \frac{aW + (1-a)\cdot G(n)\cdot W}{aW + (1-a)\cdot G(n)\cdot \frac{W}{n}} = \frac{a + (1-a)\cdot G(n)}{a + \frac{(1-a)\cdot G(n)}{n}}$$

Adding Overhead

$$S_n = \frac{T_1}{T_n} = \frac{aW + (1-a)\cdot G(n)\cdot W}{aW + \frac{(1-a)\cdot G(n)\cdot W}{n} + T_0} = \frac{a + (1-a)\cdot G(n)}{a + \frac{(1-a)\cdot G(n)}{n} + \frac{T_0}{W}}$$

Cases:

$G(n) = 1$:

$$S_n = \frac{T_1}{T_n} = \frac{aW + (1-a)\cdot 1\cdot W}{aW + (1-a)\cdot 1\cdot \frac{W}{n}} = \frac{n}{1 + (n-1)a}$$

Amdahl's Law

$G(n) = n$:

$$S_n = \frac{T_1}{T_n} = \frac{aW + (1-a)\cdot n\cdot W}{aW + (1-a)\cdot n\cdot \frac{W}{n}} = a + (1-a)n$$

Gustafson's Law

$G(n) > n$:

$$S_{n\to\infty} = \frac{T_1}{T_n} = \frac{na + n\cdot(1-a)\cdot G(n)}{na + (1-a)\cdot G(n)} \Rightarrow \frac{n\cdot(1-a)\cdot G(n)}{na + (1-a)\cdot G(n)}$$